

# Simplifying inclusion-exclusion formulas

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## Abstract

Let  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  be a family of  $n$  sets on a ground set  $X$ , such as a family of balls in  $\mathbb{R}^d$ . For every finite measure  $\mu$  on  $X$ , such that the sets of  $\mathcal{F}$  are measurable, the classical *inclusion-exclusion formula* asserts that  $\mu(F_1 \cup F_2 \cup \dots \cup F_n) = \sum_{I: \emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} F_i\right)$ ; that is, the measure of the union is expressed using measures of various intersections. The number of terms in this formula is exponential in  $n$ , and a significant amount of research, originating in applied areas, has been devoted to constructing simpler formulas for particular families  $\mathcal{F}$ . We provide the apparently first upper bound valid for an arbitrary  $\mathcal{F}$ : we show that every system  $\mathcal{F}$  of  $n$  sets with  $m$  nonempty fields in the Venn diagram admits an inclusion-exclusion formula with  $m^{O(\log^2 n)}$  terms and with  $\pm 1$  coefficients, and that such a formula can be computed in  $m^{O(\log^2 n)}$  expected time. We also construct systems of  $n$  sets on  $n$  points for which every valid inclusion-exclusion formula has the sum of absolute values of the coefficients at least  $\Omega(n^{3/2})$ .

## 1 Introduction

One of the basic topics in introductory courses of discrete mathematics is the *inclusion-exclusion principle* (also called the *sieve formula*), which allows one to compute the number of elements of a union  $F_1 \cup F_2 \cup \dots \cup F_n$  of  $n$  sets from the knowledge of the sizes of all intersections of the  $F_i$ 's.

We will consider a slightly more general setting, where we have a ground set  $S$  and a (finite) *measure*  $\mu$  on  $S$ ; then the inclusion-exclusion principle asserts that for every collection  $F_1, F_2, \dots, F_n$  of  $\mu$ -measurable sets, we have

$$\mu\left(\bigcup_{i=1}^n F_i\right) = \sum_{I: \emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} F_i\right). \quad (1)$$

(Here, as usual,  $[n] = \{1, 2, \dots, n\}$  and  $|I|$  denotes the cardinality of the set  $I$ .) This principle not only plays a fundamental role in various areas of mathematics such as probability theory or combinatorics, but it also has important algorithmic applications. For instance, it provides simple methods for the computation of volume or surface area of molecules in computational biology [PCG<sup>+</sup>92] and underlies, through efficient computation of Möbius transforms [Knu97, Section 4.3.4], the best known algorithms for several NP-hard problems including graph  $k$ -coloring [BHK09], dominating set [vRNvD09], or partial dominating set and set splitting [NvR10].

The inclusion-exclusion principle involves a number of summands that is exponential in  $n$ , the number of sets. In general this cannot be avoided if one wants an *exact* formula valid for *every* family  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ ; see Example 2.3 below for a family for which Equation (1) is the only solution. Yet, since this is a serious obstacle to efficient uses of inclusion-exclusion, much effort has been devoted to finding “smaller” formulas. These efforts essentially organize along two lines of research.

The first approach gives up on exactness and tries to *approximate* efficiently the measure of the union using the measure of only *some* of the intersections. The first results of this flavor are the classical *Bonferroni inequalities* [Bon36].<sup>1</sup> It turns out that better approximations can be obtained by replacing the coefficients  $(-1)^{|I|+1}$  by other suitable numbers, and such Bonferroni-type inequalities have been studied extensively; see, e.g., [Gal96]. Linial and Nisan [LN90] and Kahn et al. [KLS96] investigated how well  $\mu(F_1 \cup \dots \cup F_n)$  can be approximated if we know the measure of all intersections  $\bigcap_{i \in I} F_i$  for all  $I \subseteq [n]$  of size at most  $r$ . Their main finding is that having  $r$  at least of order  $\sqrt{n}$  is both necessary and sufficient for a reasonable approximation in the worst case. This still leaves us with about  $2^{\sqrt{n}}$  terms in approximate inclusion-exclusion formulas.

The second line of research looks for “small” inclusion-exclusion formulas valid for *specific* families of sets. To illustrate the type of simplifications afforded by fixing the sets, consider the family  $\mathcal{F} = \{F_1, F_2, F_3\}$  of Figure 1. Since  $F_1 \cap F_3 = F_1 \cap F_2 \cap F_3$ , Formula (1) can be simplified to

$$\mu(F_1 \cup F_2 \cup F_3) = \mu(F_1) + \mu(F_2) + \mu(F_3) - \mu(F_1 \cap F_2) - \mu(F_2 \cap F_3).$$

More generally, let us consider a family  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ , and let us say that a coefficient vector

$$\alpha = (\alpha_I)_{\emptyset \neq I \subseteq [n]} \in \mathbb{R}^{2^n - 1}$$

is an *IE-vector* for  $\mathcal{F}$  if we have

$$\mu\left(\bigcup_{i=1}^n F_i\right) = \sum_{I: \emptyset \neq I \subseteq [n]} \alpha_I \mu\left(\bigcap_{i \in I} F_i\right) \quad (2)$$

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<sup>1</sup>These assert that if we omit all terms with  $|I| > r$  on the right-hand side of (1), then we get an upper bound for the left-hand side for  $r$  odd, and a lower bound for the left-hand side for  $r$  even. The case  $r = 1$  is the often-used *union bound* in probability theory.

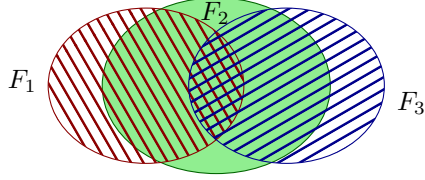


Figure 1: Three subsets of  $\mathbb{R}^2$  admitting a simpler inclusion-exclusion formula. The ground set  $F_1 \cup F_2 \cup F_3$  splits into six nonempty regions recognizable by the filling pattern.

for every finite measure  $\mu$  on the ground set of  $\mathcal{F}$  (with all the  $F_i$ 's measurable). Given  $\mathcal{F}$ , we would like to find an IE-vector for  $\mathcal{F}$ , such that both the number of nonzero coefficients is small, and the coefficients themselves are not too large. This idea, which we originally learned from [AE07], seems to originate in the work of Kratky [Kra78] on families of disks in the plane, and a systematic study of such simplifications was initiated by Naiman and Wynn [NW92, NW97]. We refer to the monograph of Dohmen [Doh03] for an overview of this line of research.

Given a specific family  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  of sets, how small can we expect an inclusion-exclusion formula to be? This is, roughly speaking, the question we tackle in this paper. To formalize the problem, we should first specify how  $\mathcal{F}$  is given. First let us consider the *Venn diagram* of  $\mathcal{F}$ , which is the partition of the ground set  $S$  into equivalence classes according to the membership in the sets of  $\mathcal{F}$ . Namely, for each nonempty index set  $\tau \subseteq [n]$ , we define the *region* of  $\tau$ , denoted by  $\text{reg}(\tau)$ , as the set of all points that belong to the sets  $F_i$  with  $i \in \tau$  and no others (see Figure 1); that is,

$$\text{reg}(\tau) = \left( \bigcap_{i \in \tau} F_i \right) \setminus \left( \bigcup_{i \notin \tau} F_i \right).$$

The *Venn diagram* of  $\mathcal{F}$  is then the collection of all subsets of  $[n]$  with non-empty regions; that is,

$$\mathcal{V} = \mathcal{V}(\mathcal{F}) := \{\tau \subseteq [n] : \text{reg}(\tau) \neq \emptyset\}.$$

Formally, we thus regard the Venn diagram as a set system on the ground set  $[n]$ ; it can be regarded as some kind of “dual” of the set system  $\mathcal{F}$ .

It is easy to see that, as far as inclusion-exclusion formulas are concerned, all points in a single region are equivalent; it only matters which of the regions are nonempty. Thus, in order to simplify our formulations, we can assume that  $\mathcal{F}$  is *standardized*, meaning that the ground set equals the union of the  $F_i$ 's and each nonempty region has exactly one point. From an algorithmic point of view, this amounts to a preprocessing step for  $\mathcal{F}$ , in which the part of the ground set  $S$  in each nonempty region is contracted to a single point.

Let  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  be a family of sets and let  $m$  denote the size of  $\mathcal{V}$  (which equals the size of the ground set for  $\mathcal{F}$  standardized). A linear-algebraic argument shows that *every* (finite) family  $\mathcal{F}$  has an inclusion-exclusion formula with at most  $m$  terms (see Corollary 2.4) and  $m$  terms are sometimes necessary (see the beginning of Section 4). The question of how small a formula  $\mathcal{F}$  admits may thus seem settled. There is, however, a caveat: this linear-algebraic argument may yield *exponentially large* coefficients (see Example 2.5). If we wanted to use such a formula, we would need to compute with very high precision, and perhaps more seriously, we would have to know the measures of the various intersections with an enormous precision, in order to obtain a

meaningful result. This may be totally impractical, e.g., in geometric settings where some physical measurements are involved, or where the measures of the intersections are computed with limited precision. Thus, we prefer inclusion-exclusion formulas where not only the number of terms is small, but the coefficients are also small.

Our main result is the following general upper bound; to our knowledge, it is the first upper bound applicable for an arbitrary family.

**Theorem 1.1.** *Let  $n$  and  $m$  be integers and let  $D = \lceil 2e \ln m \rceil \lceil 1 + \ln \frac{n}{\ln m} \rceil$ . Then for every family  $\mathcal{F}$  of  $n$  sets with Venn diagram of size  $m$ , there is an IE-vector  $\alpha$  for  $\mathcal{F}$  that has at most  $\sum_{i=1}^D \binom{n}{i} \leq m^{O(\ln^2 n)}$  nonzero coefficients, and in which all nonzero coefficients are  $\pm 1$ 's. Such an  $\alpha$  can be computed in  $m^{O(\ln^2 n)}$  expected time if  $\mathcal{F}$  is standardized.*

The bound in this theorem is pseudo-polynomial, but not polynomial, in  $m$  and  $n$ . We do not know if a polynomial bound can be achieved, for example, with  $\pm 1$  coefficients. We have at least the following lower bound, proved in Section 4, showing that inclusion-exclusion formulas of *linear* size are impossible in general.

**Theorem 1.2.** *For infinitely many values of  $n$ , there are families of  $n$  sets on  $n$  points, for which every IE-vector has  $\ell_1$ -norm at least  $(n/2)^{3/2}$ .*

We recall that the  $\ell_1$ -norm of a real vector  $\mathbf{x} \in \mathbb{R}^d$  is  $\|\mathbf{x}\|_1 = \sum_{i=1}^d |x_i|$ . The  $\ell_1$ -norm gives a lower bound on the tradeoff between the number of nonzero coefficients and their orders of magnitude (we recall that a formula with  $O(m)$  nonzero coefficients is always attainable, the problem being that the coefficients may be too large).

**Remark on  $\ell_1$ -norm minimization.** A useful heuristic for finding “small” IE-vectors might be to look for an IE-vector of minimum  $\ell_1$ -norm. In the linear-algebraic formulation, this means finding a solution of  $A\mathbf{x} = \mathbf{1}$  of minimum  $\ell_1$ -norm.

It is well known that finding a solution of minimum  $\ell_1$ -norm of a linear system can be done in polynomial time, via linear programming. Several specialized algorithms for this problem have also been developed, with better performance than direct application of general-purpose LP solvers (see, e.g., [YGZ<sup>+</sup>10] for a recent overview). However, in our setting the number of columns of the matrix  $A$  may be exponential in  $m$  and  $n$ , and so even the input for an  $\ell_1$ -norm minimizing algorithm would be too large.

There are linear programs with exponentially many variables (and polynomially many constraints) that can still be solved in polynomial time. For example, one may attempt, at least for theoretical purposes, to solve the dual linear program by the ellipsoid method, provided that a separation oracle is available.

In our setting, the task of the separation oracle can be formulated as follows in the setting of the original (standardized) set system  $\mathcal{F} = \{F_1, \dots, F_n\}$ : *Given weights  $w_1, \dots, w_m \in \mathbb{Z}$  of the points and threshold  $c$ , find a subset  $I \subseteq [n]$ , if one exists, such that the sum of weights of the points in  $\bigcap_{i \in I} F_i$  is at least  $c$ .* Unfortunately, as was shown by Hoffmann et al. [HOR<sup>+</sup>12], this problem is NP-complete not only for arbitrary set systems, but also, e.g., for the case where each  $F_i$  is the complement of a hexagon in the plane. Thus, this approach doesn't seem to lead to a polynomial-time algorithm for finding an IE-vector of minimum  $\ell_1$ -norm even for rather simple geometric settings.

## 2 Preliminaries

As in the introduction, we consider a family  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  of sets on a ground set  $S$ , and we assume that the  $F_i$  are all distinct. Besides the Venn diagram  $\mathcal{V}$ , we associate yet another set system with  $\mathcal{F}$ , namely, the *nerve*  $\mathcal{N}$  of  $\mathcal{F}$ :

$$\mathcal{N} = \mathcal{N}(\mathcal{F}) := \left\{ \sigma \subseteq [n] : \sigma \neq \emptyset, \bigcap_{i \in \sigma} F_i \neq \emptyset \right\}.$$

So both of  $\mathcal{N}$  and  $\mathcal{V}$  have ground set  $[n]$ , and we have  $\mathcal{V} \subseteq \mathcal{N}$ .

Let us enumerate the elements of  $\mathcal{V}$  as  $\mathcal{V} = \{\tau_1, \tau_2, \dots, \tau_m\}$  in such a way that  $|\tau_i| \leq |\tau_j|$  for  $i < j$ , and let us enumerate  $\mathcal{N} = \{\sigma_1, \sigma_2, \dots, \sigma_{|\mathcal{N}|}\}$  so that the sets of  $\mathcal{V}$  come first, i.e.,  $\sigma_i = \tau_i$  for  $i = 1, 2, \dots, m$ .

In the introduction, we were indexing IE-vectors for  $\mathcal{F}$  by all possible subsets  $I \subseteq [n]$ . But if  $I$  is not in the nerve, the corresponding intersection is empty, and thus w.l.o.g. we may assume that its coefficient is zero. Thus, from now on, we will index IE-vectors  $\mathbf{x}$  as  $(x_1, \dots, x_{|\mathcal{N}|})$ , where  $x_j$  is the coefficient of  $\mu(\bigcap_{i \in \sigma_j} F_i)$ .

**IE-vectors from linear algebra.** Let  $A = (a_{jk})$  denote the 0-1 matrix with  $m$  rows and  $|\mathcal{N}|$  columns such that  $a_{jk} = 1$  if  $\tau_j \supseteq \sigma_k$  and  $a_{jk} = 0$  otherwise. Let  $\mathbf{1}$  denote the  $m$ -dimensional vector with all entries equal to 1.

**Lemma 2.1.**  $\mathbf{x} \in \mathbb{R}^{|\mathcal{N}|}$  is an IE-vector for  $\mathcal{F}$  if and only if  $A\mathbf{x} = \mathbf{1}$ .

*Proof.* A vector  $\mathbf{x} \in \mathbb{R}^{|\mathcal{N}|}$  is an IE-vector for  $\mathcal{F}$  if and only if for every finite measure  $\mu$  on  $S$  we have

$$\mu\left(\bigcup_{i=1}^n F_i\right) = \sum_{k=1}^{|\mathcal{N}|} x_k \mu\left(\bigcap_{i \in \sigma_k} F_i\right). \quad (3)$$

We first reformulate Equation (3) using the regions of  $\mathcal{F}$ . The regions decompose  $\bigcup_{i=1}^n F_i$  in a way that is compatible with the regions  $\bigcap_{i \in \sigma} F_i$ :

$$\bigcup_{i=1}^n F_i = \bigcup_{\tau \in \mathcal{V}} \text{reg}(\tau) \quad \text{and for all } \sigma \in \mathcal{N}, \quad \bigcap_{i \in \sigma} F_i = \bigcup_{\tau \in \mathcal{V}: \tau \supseteq \sigma} \text{reg}(\tau).$$

Moreover, the regions are pairwise disjoint. Thus, for every finite measure  $\mu$  on  $S$  we have

$$\mu\left(\bigcup_{i=1}^n F_i\right) = \sum_{\tau \in \mathcal{V}} \mu(\text{reg}(\tau)) \quad \text{and for all } \sigma \in \mathcal{N}, \quad \mu\left(\bigcap_{i \in \sigma} F_i\right) = \sum_{\tau \in \mathcal{V}: \tau \supseteq \sigma} \mu(\text{reg}(\tau)),$$

and Equation (3) is equivalent to

$$\sum_{\tau \in \mathcal{V}} \mu(\text{reg}(\tau)) = \sum_{k=1}^{|\mathcal{N}|} x_k \left( \sum_{\tau \in \mathcal{V}: \tau \supseteq \sigma_k} \mu(\text{reg}(\tau)) \right).$$

Using the orderings on  $\mathcal{V}$  and  $\mathcal{N}$  and the definition of  $A$  we obtain that  $\mathbf{x} \in \mathbb{R}^{|\mathcal{N}|}$  is an IE-vector for  $\mathcal{F}$  if and only if for every finite measure  $\mu$  on  $S$  we have

$$\sum_{j=1}^m \mu(\text{reg}(\tau_j)) = \sum_{k=1}^{|\mathcal{N}|} x_k \left( \sum_{j=1}^m a_{j,k} \mu(\text{reg}(\tau_j)) \right) = \sum_{j=1}^m \left( \sum_{k=1}^{|\mathcal{N}|} a_{j,k} x_k \right) \mu(\text{reg}(\tau_j)). \quad (4)$$

Now, if  $A\mathbf{x} = \mathbf{1}$  then Equation (4) trivially holds for all  $\mu$  and  $\mathbf{x}$  is an IE-vector for  $\mathcal{F}$ . Conversely, assume that  $\mathbf{x}$  is an IE-vector for  $\mathcal{F}$  and thus that Equation (4) holds for all  $\mu$ . For  $1 \leq j \leq m$  we pick  $p_j \in \text{reg}(\tau_j)$  and define the measure  $\mu_j : 2^S \rightarrow \mathbb{R}$  by  $\mu_j(T) = 1$  if  $p_j \in T$  and 0 otherwise. Equation (4) then specializes to

$$1 = \mu_j(\text{reg}(\tau_j)) = \sum_{k=1}^{|\mathcal{N}|} x_k a_{j,k} \mu_j(\text{reg}(\tau_j)) = \sum_{k=1}^{|\mathcal{N}|} a_{j,k} x_k.$$

This implies that  $(A\mathbf{x})_j = 1$ . The statement follows.  $\square$

*Remark 2.2.* In our definition a vector  $\mathbf{x}$  is an IE-vector for  $\mathcal{F}$  if and only if Equation (2) is valid for every finite measure. As it follows from the proof of Lemma 2.1 this definition is equivalent to extending this requirement to every (finitely additive) signed measure. (A signed measure satisfies the classical axioms of a measure with the exception that it may take negative values.)

*Example 2.3.* Let  $S = 2^{[n]} \setminus \{[n]\}$  and  $F_i = 2^{[n] \setminus \{i\}}$  for  $i \in [n]$ . It is easy to see that here  $\mathcal{N} = \mathcal{V}$  and  $A$  is a lower-triangular square matrix with 1's on the diagonal. Hence  $A$  is invertible and, by Lemma 2.1,  $\mathcal{F}$  has a unique IE-vector, namely, the one from the standard inclusion-exclusion formula.

**Corollary 2.4.** *For every finite family  $\mathcal{F}$ , there is a unique IE-vector  $\boldsymbol{\alpha}$  supported on  $\mathcal{V}$  (that is, such that  $\alpha_I = 0$  for  $I \notin \mathcal{V}$ ), and this  $\boldsymbol{\alpha}$  has all entries integral.*

*Proof.* Let  $B$  be the  $m \times m$  submatrix of  $A$  consisting of the first  $m$  columns of  $A$ . The IE-vectors for  $\mathcal{F}$  supported on  $\mathcal{V}$  are in one-to-one correspondence with the solutions of  $B\mathbf{y} = \mathbf{1}$ . Since  $B$  is lower-triangular and has 1's on the main diagonal, it is nonsingular, and hence  $B\mathbf{y} = \mathbf{1}$  has exactly one solution. Moreover, since  $B$  is a lower-triangular 0-1 matrix, this solution is integral.  $\square$

Unfortunately, the IE-vector with small support given by Corollary 2.4 might have exponentially large coefficients, as the following example shows.

*Example 2.5.* Let  $S = [5\ell]$  for some positive integer  $\ell$ , and for  $i \leq \ell$ , let  $g(i)$  stand for the smallest integer  $j \geq i$  divisible by 5; that is  $g(i) = 5\lceil i/5 \rceil$ . We consider the set system  $\mathcal{F} = \{F_1, F_2, \dots, F_{5\ell}\}$  on  $S$  given by  $F_i = \{i\} \cup \{g(i)+1, \dots, 5\ell\}$ . Now  $j \in F_i$  if and only if  $i = j$  or  $j > g(i)$ . In particular, no two elements of  $S$  belong to the same region and the number of regions of  $\mathcal{F}$  is  $m = |S| = 5\ell$ , which is also equal to the number  $n$  of sets in  $\mathcal{F}$ :  $n = m = 5\ell$ . The lower-triangular matrix  $B$  from the proof of Corollary 2.4 has a simple structure in terms of  $5 \times 5$  blocks: the blocks on the diagonal are identity blocks, and the blocks below the diagonal are filled with 1's. Let  $\hat{\mathbf{x}}$  denote the solution of  $B\mathbf{x} = \mathbf{1}$ . The first five rows yield  $\hat{x}_1 = \hat{x}_2 = \dots = \hat{x}_5 = 1$ . The next five rows imply that for  $j = 6, 7, \dots, 10$  we have

$$\hat{x}_1 + \hat{x}_2 + \dots + \hat{x}_5 + \hat{x}_j = 1,$$

and so  $\hat{x}_6 = \hat{x}_7 = \dots = \hat{x}_{10} = -4$ . A simple induction yields  $\hat{x}_i = (-4)^{(g(i)/5)-1}$ . Altogether, the largest coefficient is of order  $4^{n/5}$ . (Replacing the constant 5 by another constant  $y$  yields a similar exponential growth with basis  $(y-1)^{1/y}$ ; the choice  $y = 5$  maximizes the basis of the exponent.)

**Abstract tubes.** While studying possible simplifications of inclusion-exclusion formulas, Naiman and Wynn [NW92, NW97] started from families  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  that were tube-like in the sense that  $F_i \cap F_j \subseteq F_k$  for all  $i \leq k \leq j$  (as in our Figure 1). The simplifications identified for “simple tubes” hold in a broader setting, leading Naiman and Wynn to introduce the more general notion of an abstract tube. This notion will also play an important role in our considerations.

**Definition 2.6.** An *(abstract) simplicial complex* with vertex set  $[n]$  is a hereditary system of nonempty subsets of  $[n]$ .<sup>2</sup> An *abstract tube* is a pair  $(\mathcal{F}, \mathcal{K})$ , where  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  is a family of sets and  $\mathcal{K}$  is a simplicial complex with vertex set  $[n]$ , such that for every nonempty region  $\tau$  of the Venn diagram of  $\mathcal{F}$ , the subcomplex *induced* on  $\mathcal{K}$  by  $\tau$ ,  $\mathcal{K}[\tau] := \{\vartheta \in \mathcal{K} : \vartheta \subseteq \tau\}$ , is contractible.<sup>3</sup>

As first noted by Naiman and Wynn [NW92, NW97], if  $(\mathcal{F}, \mathcal{K})$  is an abstract tube, then

$$\mu\left(\bigcup_{i=1}^n F_i\right) = \sum_{I \in \mathcal{K}} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} F_i\right). \quad (5)$$

Moreover, truncating the sum yields upper and lower bounds in the spirit of the Bonferroni inequalities [Doh03, Theorem 3.1.9].

*Remark 2.7.* An earlier, more permissive definition of abstract tubes by [NW92] had the weaker condition “ $\chi(\mathcal{K}[\tau]) = 1$ ” instead of “ $\mathcal{K}[\tau]$  contractible,” where  $\chi$  is the *Euler characteristic*. We recall that for a simplicial complex  $\mathcal{L}$  in our sense, we have  $\chi(\mathcal{L}) = \sum_{\sigma \in \mathcal{L}} (-1)^{|\sigma|+1}$ . Using this definition and Lemma 2.1, the proof of (5) can be given in few lines. Indeed, consider a simplicial complex  $\mathcal{K}$  with vertex set  $[n]$  and let  $\mathbf{x} \in \mathbb{R}^{[N]}$  stand for the vector with  $x_k = (-1)^{|\sigma_k|+1}$  if  $\sigma_k \in \mathcal{K}$  and  $x_k = 0$  otherwise. Since

$$(A\mathbf{x})_j = \sum_{k: \sigma_k \subseteq \tau_j} x_k = \sum_{\sigma_k: \sigma_k \in \mathcal{K}[\tau_j]} (-1)^{|\sigma_k|+1},$$

we have  $(A\mathbf{x})_j = \chi(\mathcal{K}[\tau_j])$ . Thus, if all the  $\mathcal{K}[\tau_j]$  have Euler characteristic 1, then  $\mathbf{x}$  is an IE-vector, and (5) follows.

The stronger definition of abstract tubes involving contractibility was needed in order to guarantee that truncations of Equation (5) also yield Bonferroni-type inequalities [Doh03, Theorem 3.1.9].

Small abstract tubes have been identified for families of balls [NW92, NW97, AE07] or half-spaces [NW97] in  $\mathbb{R}^d$ , and similar structures were found for families of pseudodisks [ER97]. We establish Theorem 1.1 by proving that for every family of sets there exists an abstract tube with “small” size that, in addition, can be computed efficiently. We will use the following sufficient

<sup>2</sup>We emphasize that we exclude an empty set from the definition of a simplicial complex. This is non-standard definition; however, it is convenient for our purposes.

<sup>3</sup>By *contractible* we mean contractibility in the sense of topology; roughly speaking, the topological space defined by  $\mathcal{K}[\tau]$  can be continuously shrunk to a point. Readers not at ease with this notion may want to look at the remark few lines below.

condition guaranteeing that  $(\mathcal{F}, \mathcal{K})$  is an abstract tube; it is a reformulation of [Doh03, Theorem 4.2.5] (for the reader's convenience we include a simple proof). Let  $\text{MNF}(\mathcal{K})$  denote the system of all inclusion-minimal non-faces of  $\mathcal{K}$ , i.e., of all nonempty sets  $I \subseteq [n]$  with  $I \notin \mathcal{K}$  but with  $I' \in \mathcal{K}$  for every proper subset  $I' \subset I$ .

**Proposition 2.8.** *Let  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  be a family of sets with Venn diagram  $\mathcal{V}$  and let  $\mathcal{K}$  be a simplicial complex with vertex set  $[n]$ . If no set of  $\mathcal{V}$  can be expressed as a union of sets in  $\text{MNF}(\mathcal{K})$ , then  $(\mathcal{F}, \mathcal{K})$  is an abstract tube.*

*Proof.* Let  $\tau \in \mathcal{V}$  and let  $a \in \tau$  such that  $a$  belongs to no element of  $\text{MNF}(\mathcal{K})$  contained in  $\tau$ . For every simplex  $\vartheta \in \mathcal{K}[\tau]$ , we have  $\vartheta \cup \{a\} \in \mathcal{K}[\tau]$ . If  $\vartheta \cup \{a\} \notin \mathcal{K}[\tau]$ , then  $\vartheta \cup \{a\}$  contains some  $\beta \in \text{MNF}(\mathcal{K})$ ; as  $\vartheta \in \mathcal{K}[\tau]$ , it must be that  $\beta$  contains  $a$ , a contradiction. Thus  $\vartheta \cup \{a\} \in \mathcal{K}[\tau]$  for every  $\vartheta \in \mathcal{K}[\tau]$ . In other words,  $\mathcal{K}[\tau]$  is a cone with apex  $a$ . Since every cone is contractible, the statement follows.  $\square$

### 3 The upper bound: proof of Theorem 1.1

**Abstract tubes from selectors.** Let  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  be a family of sets, and let  $\mathcal{V}$  be the Venn diagram of  $\mathcal{F}$ . We recall that a *selector* for  $\mathcal{V}$  is a map  $w: \mathcal{V} \rightarrow [n]$  such that  $w(\tau) \in \tau$  for every  $\tau \in \mathcal{V}$ . We observe that each selector for  $\mathcal{V}$  provides an abstract tube for  $\mathcal{F}$  (which satisfies the sufficient condition of Proposition 2.8).

**Lemma 3.1.** *Let  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ ,  $\mathcal{V} = \mathcal{V}(\mathcal{F})$ , and let  $w$  be a selector for  $\mathcal{V}$ . We define the simplicial complex*

$$\mathcal{K}_w = \{\sigma \in \mathcal{N}(\mathcal{F}) : \text{for all nonempty } \vartheta \subseteq \sigma \text{ there is } \tau \in \mathcal{V} \text{ such that } w(\tau) \in \vartheta \subseteq \tau\}.$$

*Then  $(\mathcal{F}, \mathcal{K}_w)$  is an abstract tube.*

*Proof.* This is simple once the idea behind the definition of  $\mathcal{K}_w$  is explained. Namely, in the condition of Proposition 2.8 we want to prevent each set  $\tau \in \mathcal{V}$  from being a union of minimal non-faces of the simplicial complex  $\mathcal{K}$ . Our way of achieving that is to insist that every minimal non-face  $I$  contained in  $\tau$  avoids the point  $w(\tau)$ ; thus, we consider the set system of “admissible minimal non-faces”

$$\mathcal{B}_w := \{I \subseteq [n], I \neq \emptyset : \text{if } I \subseteq \tau \in \mathcal{V}, \text{ then } w(\tau) \notin I\}.$$

Then the above definition of  $\mathcal{K}_w$  can be interpreted as follows: a simplex  $\sigma \in \mathcal{N}$  belongs to  $\mathcal{K}_w$  if it contains no  $I \in \mathcal{B}_w$ .<sup>4</sup> (Simplexes outside  $\mathcal{N}$  can be ignored, since their supersets cannot be contained in a set  $\tau \in \mathcal{V}$ .) Therefore, all minimal non-faces of  $\mathcal{K}_w$  belong to  $\mathcal{B}_w$  or lie outside  $\mathcal{N}$ , and hence  $(\mathcal{F}, \mathcal{K}_w)$  is an abstract tube by Proposition 2.8.  $\square$

Let us remark that there is no loss of generality in passing from the abstract tubes as in Proposition 2.8 to those of the form  $\mathcal{K}_w$ . Indeed, if  $\mathcal{K}$  satisfies the condition of Proposition 2.8, then every  $\tau \in \mathcal{V}$  contains at least one point that is not contained in any minimal non-face  $I$  of  $\mathcal{K}$  with  $I \subseteq \tau$ , and such a point can be chosen as  $w(\tau)$ —then we can easily check that  $\mathcal{K}_w \subseteq \mathcal{K}$ .

<sup>4</sup>Note that for the formal verification, the condition  $\sigma$  contains no  $I \in \mathcal{B}_w$  can be written, in symbols, as follows:  $\forall I \subseteq [n], I \neq \emptyset : ((\forall \tau \in \mathcal{V} : I \subseteq \tau \Rightarrow w(\tau) \notin I) \Rightarrow I \not\subseteq \sigma)$ . This is equivalent to  $\forall I \subseteq [n], I \neq \emptyset : I \subseteq \sigma \Rightarrow (\exists \tau \in \mathcal{V} : I \subseteq \tau \wedge w(\tau) \in I)$  which is just a transcription of  $\sigma \in \mathcal{K}_w$ .



		$i_1$	$i_2$	$i_3$		$i_4$		$i_5$	
$\vdots$									
$j_3$	$0 \cdots 0$	0	0	1	?	1	?	1	?
$\vdots$									
$j_1$	$0 \cdots 0$	1	1	1	?	1	?	1	?
$j_2$	$0 \cdots 0$	0	1	1	?	1	?	1	?
$j_4$	$0 \cdots 0$	0	0	0	$0 \cdots 0$	1	?	1	?
$j_5$	$0 \cdots 0$	0	0	0	$0 \cdots 0$	0	$0 \cdots 0$	1	?
$\vdots$									

Figure 2: Illustration for Lemma 3.2: If  $\mathcal{K}_\rho$  contains the simplex  $\{i_1, i_2, \dots, i_5\}$ , then  $\Gamma_\rho$  must contain a row  $j_s$  compatible with  $\{i_s, i_{s+1}, \dots, i_5\}$  for  $s = 1, 2, \dots, 5$ . The  $j_3$  row is emphasized, constrained values appearing in grey; rows  $j_s$  for other values of  $s$  are represented consecutively for clarity, but they can appear in any order and non-consecutively.

**Large simplices in random  $\mathcal{K}_w$ .** Let  $\rho$  be a permutation of  $[n]$ . We define a selector  $w_\rho$  for  $\mathcal{V}$  by taking  $w(\tau)$  as the smallest element of  $\tau$  in the linear ordering  $\prec$  on  $[n]$  given by  $\rho(1) \prec \rho(2) \prec \dots \prec \rho(n)$ .

For better readability we write  $\mathcal{K}_\rho$  instead of  $\mathcal{K}_{w_\rho}$ . We want to show that for random  $\rho$ ,  $\mathcal{K}_\rho$  is unlikely to contain too large simplices, and thus leads to a small inclusion-exclusion formula.

Let  $\Gamma$  denote the incidence matrix of  $\mathcal{V}$ , that is, the 0-1 matrix with  $m$  rows and  $n$  columns where  $\Gamma_{ij} = 1$  if and only if  $j \in \tau_i$  (if the original system  $\mathcal{F}$  was standardized, then  $\Gamma$  is the transposition of the usual incidence matrix of  $\mathcal{F}$ ). We also denote by  $\Gamma_\rho$  the matrix obtained by applying the permutation  $\rho$  to the columns of  $\Gamma$ : the  $\rho(i)$ th column of  $\Gamma_\rho$  is the  $i$ th column of  $\Gamma$  and represents the incidences between permuted  $[n]$  and  $\mathcal{V}$ . We now argue that if  $\mathcal{K}_\rho$  contains a large simplex, then  $\Gamma_\rho$  contains a particular substructure.

We say that a row  $R$  of  $\Gamma_\rho$  is *compatible* with a subset  $I \subseteq [n]$  if  $R$  contains 1's in all columns with index in  $I$  and 0's in all columns with index smaller than  $\min(I)$ .

**Lemma 3.2.** *If  $\rho(\tau) = \{i_1, i_2, \dots, i_k\}$  for a simplex  $\tau$  in  $\mathcal{K}_\rho$ , with  $i_1 < i_2 < \dots < i_k$ , then for every  $s \in \{1, 2, \dots, k\}$  the matrix  $\Gamma_\rho$  contains a row compatible with  $\{i_s, i_{s+1}, \dots, i_k\}$ .*

*Proof.* Let  $s \in \{1, 2, \dots, k\}$ , let  $I_s = \{i_s, i_{s+1}, \dots, i_k\}$ , and let  $\vartheta_s = \rho^{-1}(I_s)$ . We refer to Figure 2. Since  $\vartheta_s$  is a simplex of  $\mathcal{K}_\rho$ , there exists  $\tau_{j_s} \in \mathcal{V}$  such that  $w_\rho(\tau_{j_s}) \in \vartheta_s \subseteq \tau_{j_s}$  by Lemma 3.1. Since  $\vartheta_s \subseteq \tau_{j_s}$ , we have  $I_s = \rho(\vartheta_s) \subseteq \rho(\tau_{j_s})$ , and hence the  $j_s$ th row of  $\Gamma_\rho$  has 1's in all columns with index in  $I_s$ . Since  $w_\rho(\tau_{j_s}) \in \vartheta_s$ , the set  $\rho(\tau_{j_s})$  contains no  $i$  with  $i < i_s$  and the  $j_s$ th row of  $\Gamma_\rho$  has 0's in all columns with index smaller than  $i_s = \min(I_s)$ . It follows that the  $j_s$ th row of  $\Gamma_\rho$  is compatible with  $I_s$ .  $\square$

We will need the following inequality.

**Lemma 3.3.** *Let  $x_1, \dots, x_r$  be positive real numbers with  $x_1 + \dots + x_r \leq n$ . Then*

$$\frac{x_1}{x_1 + \dots + x_r} \cdot \frac{x_2}{x_2 + \dots + x_r} \cdots \frac{x_{r-1}}{x_{r-1} + x_r} \leq \left(1 - \sqrt[r-1]{\frac{x_r}{n}}\right)^{r-1}.$$

*Proof.* Let us set  $y_\ell := x_\ell + x_{\ell+1} + \dots + x_r$ . Then we have

$$\begin{aligned}
\frac{x_1}{x_1 + \dots + x_r} \cdot \frac{x_2}{x_2 + \dots + x_r} \dots \frac{x_{r-1}}{x_{r-1} + x_r} &= \frac{y_1 - y_2}{y_1} \cdot \frac{y_2 - y_3}{y_2} \dots \frac{y_{r-1} - y_r}{y_{r-1}} \\
&= \left(1 - \frac{y_2}{y_1}\right) \cdot \left(1 - \frac{y_3}{y_2}\right) \dots \left(1 - \frac{y_r}{y_{r-1}}\right) \\
&\leq \left(\frac{1 - y_2/y_1 + 1 - y_3/y_2 + \dots + 1 - y_r/y_{r-1}}{r-1}\right)^{r-1} \\
&= \left(1 - \frac{y_2/y_1 + y_3/y_2 + \dots + y_r/y_{r-1}}{r-1}\right)^{r-1} \\
&\leq \left(1 - \sqrt[r-1]{\frac{y_r}{y_1}}\right)^{r-1} \\
&\leq \left(1 - \sqrt[r-1]{\frac{x_r}{n}}\right)^{r-1}.
\end{aligned}$$

□

Now we aim at showing that for a random  $\rho$ , the condition in Lemma 3.2 is unlikely to be satisfied for large  $k$ . That condition prescribes the existence of  $k$  rows in  $\Gamma_\rho$  with a certain pattern. In order to get a good bound for  $k$ , we won't actually look for all of these  $k$  rows, but rather we will consider only each  $b$ th of them, for a suitable integer parameter  $b$ , and ignore the rest.

Namely, we fix two parameters  $r$  and  $b$  with  $1 < b < n$  and set  $k = rb$  (we think of  $r \approx \ln n$  and  $b \approx \ln m$ ). For an  $r$ -element index set  $J \subseteq [m]$ , let  $\Gamma_\rho[J]$  denote the submatrix obtained from  $\Gamma_\rho$  by considering only the rows with indices in  $J$ . We say that a permutation  $\rho$  is *bad* for  $J$  if there is a  $k$ -element set of column indices  $I = \{i_1, i_2, \dots, i_k\}$  with  $i_1 < i_2 < \dots < i_k$  such that for every  $s \in \{1, b+1, \dots, (r-1)b+1\}$ , the matrix  $\Gamma_\rho[J]$  contains a row compatible with  $\{i_s, i_{s+1}, \dots, i_k\}$ . Finally, we define  $p_J$  as the probability that a random permutation  $\rho$  is bad for  $J$ .

**Lemma 3.4.** *We have  $p_J \leq (1 - (b/n)^{1/(r-1)})^{b(r-1)}$ .*

*Proof.* Let  $\rho$  be a bad permutation for  $J$ , and let  $I = \{i_s, i_{s+1}, \dots, i_k\}$  be the corresponding set of column indices.

Let  $\ell \in \{0, 1, \dots, r-1\}$ . There are  $r - \ell$  rows of  $\Gamma_\rho[J]$  that are compatible with a subset of  $\{i_{\ell \cdot b+1}, i_{\ell \cdot b+2}, \dots, i_k\}$ . It follows that for  $i \leq i_{\ell \cdot b+1}$ , the  $i$ th column of  $\Gamma_\rho[J]$  contains at most  $\ell$  entries 1. Moreover, for  $i \in \{i_{\ell \cdot b+1}, i_{\ell \cdot b+2}, \dots, i_{(\ell+1) \cdot b}\}$ , the  $i$ th column of  $\Gamma_\rho[J]$  contains exactly  $\ell$  entries 1, since every row compatible with  $\{i_s, i_{s+1}, \dots, i_k\}$  for  $s \leq \ell \cdot b + 1$  has an entry 1 in these columns.

We now partition  $[n]$  into  $[n] = Q_0 \cup Q_1 \cup \dots \cup Q_r$ , where  $Q_\ell$  consists of the indices of those columns of  $\Gamma[J]$  that contain exactly  $\ell$  entries 1 (and  $r - \ell$  entries 0). For  $\ell \in [r]$  and  $p \in [b]$ , let  $g_\ell^{(p)}$  denote the  $p$ th smallest element of  $\rho(Q_\ell)$ . A necessary condition on  $\rho$  is

$$g_1^{(b)} < g_2^{(1)} < g_2^{(b)} < g_3^{(1)} < \dots < g_{r-1}^{(b)} < g_r^{(1)}.$$

For  $\ell \in [r]$ , let  $E_\ell$  denote the event  $E_\ell := \{g_\ell^{(b)} < \min(g_{\ell+1}^{(1)}, g_{\ell+2}^{(1)}, \dots, g_r^{(1)})\}$ , and we bound  $p_J$  by the conditional probability

$$p_J \leq \text{Prob}(E_1) \cdot \text{Prob}(E_2|E_1) \cdot \text{Prob}(E_3|E_1 \cap E_2) \dots \text{Prob}(E_{r-1}|E_1 \cap \dots \cap E_{r-2}). \quad (6)$$

For  $\ell \in [r-1]$ ,  $\text{Prob}(E_\ell | E_1 \cap \dots \cap E_{\ell-1})$  is the probability that the  $b$  smallest elements of  $\rho(Q_\ell) \cup \rho(Q_{\ell+1}) \cup \dots \cup \rho(Q_r)$  belong to  $\rho(Q_\ell)$ . This probability is equal to

$$\binom{|Q_\ell|}{b} \bigg/ \binom{|Q_\ell| + |Q_{\ell+1}| + \dots + |Q_r|}{b} \leq \left( \frac{|Q_\ell|}{|Q_\ell| + |Q_{\ell+1}| + \dots + |Q_r|} \right)^b.$$

So, letting  $x_\ell = |Q_\ell|$ , Inequality (6) implies

$$p_J \leq \left( \frac{x_1}{x_1 + x_2 + \dots + x_r} \cdot \frac{x_2}{x_2 + x_3 + \dots + x_r} \cdot \dots \cdot \frac{x_{r-1}}{x_{r-1} + x_r} \right)^b \leq \left( 1 - \sqrt[r-1]{\frac{|Q_r|}{n}} \right)^{b(r-1)},$$

the last inequality being Lemma 3.3. Then the lemma follows using  $|Q_r| \geq b$ .  $\square$

*Proof of Theorem 1.1.* Let  $n$  and  $m \geq 3$  be integers. Let  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  be a family of  $n$  sets whose Venn diagram  $\mathcal{V}$  has size  $m$ . Let  $p(k)$  denote the probability that  $\mathcal{K}_\rho$  contains at least one simplex of size  $k$ , where  $\rho$  is chosen uniformly at random among all permutations of  $[n]$ . From Lemmas 3.2 and 3.4, for every  $r > 2$  and  $b \geq 2$  we have

$$p(rb) \leq \binom{m}{r} \left( 1 - \sqrt[r-1]{b/n} \right)^{b(r-1)} \leq m^r e^{b(r-1) \ln(1 - \sqrt[r-1]{b/n})} \leq m^r e^{-b(r-1) \sqrt[r-1]{b/n}}.$$

Assuming that  $b \geq 2e \ln m$ , we get  $p(rb) \leq m^{r-2e(r-1) \sqrt[r-1]{b/n}}$ , and choosing  $r \geq 1 + \ln \frac{n}{b}$ , we obtain

$$\sqrt[r-1]{b/n} = e^{-\frac{1}{r-1} \ln \frac{n}{b}} \geq e^{-1} \quad \text{and} \quad p(rb) \leq m^{2-r} \leq \frac{1}{2}.$$

Thus, with  $D = \lceil 2e \ln m \rceil \lceil 1 + \ln \frac{n}{\ln m} \rceil$  as in the theorem, we have  $p(D) \leq \frac{1}{2}$ , and so there exists a permutation  $\rho^*$  of  $[n]$  such that  $\mathcal{K}_{\rho^*}$  contains no simplex of size  $D$  (or larger). By Lemma 3.1,  $(\mathcal{F}, \mathcal{K}_{\rho^*})$  is an abstract tube and  $\mathcal{K}_{\rho^*}$  has at most  $\sum_{i=1}^D \binom{n}{i}$  simplices. The IE-vector obtained from the abstract tube  $(\mathcal{F}, \mathcal{K}_{\rho^*})$  as in Equation (5) is as claimed in the theorem.

In order to actually compute a suitable coefficient vector, we choose a random permutation  $\rho$  and compute  $\mathcal{K}_\rho$  by the following incremental algorithm. We use two auxiliary set systems  $\mathcal{A}$  and  $\mathcal{B}$ , initialized to  $\mathcal{A} = \mathcal{B} = \{\emptyset\}$  (the idea is that  $\mathcal{B}$  contains all the simplices of  $\mathcal{K}_\rho$  found so far, and  $\mathcal{A} \subseteq \mathcal{B}$  contains those for which we still need to test one-element extensions). In each step, we take some  $\sigma \in \mathcal{A}$ , remove it from  $\mathcal{A}$ , and for each  $i \notin \sigma$ , we test whether  $\sigma \cup \{i\}$  belongs to  $\mathcal{K}_\rho$  (for this, we just check if there is  $\tau \in \mathcal{V}$  such that  $w_\rho(\tau) \in \sigma \cup \{i\} \subseteq \tau$ ). Those  $\sigma \cup \{i\}$  that pass this test are added to both  $\mathcal{A}$  and  $\mathcal{B}$ . The algorithm finishes either when  $\mathcal{A} = \emptyset$  (in this case we set  $\mathcal{K}_\rho = \mathcal{B} \setminus \{\emptyset\}$  and return the corresponding IE-vector), or when we first discover a simplex  $\sigma \in \mathcal{K}_\rho$  of size larger than  $D$ . In the latter case, we discard the current permutation  $\rho$ , choose a new one, and repeat the algorithm.

The choice of a random permutation  $\rho$  takes  $O(n \ln n)$  time and  $n$  random bits. Accepting or rejecting a new simplex by brute-force testing takes  $O(mn)$  time. The expected number of times we have to start over with a new permutation  $\rho$  is  $O(1)$ . Altogether, the expected running time of this algorithm is  $O\left(\binom{n}{D} mn\right) = m^{O(\ln^2 n)}$ .  $\square$

## 4 The lower bound: proof of Theorem 1.2

For any  $m$  between  $n$  and  $2^n$  there exists a system of  $n$  sets with Venn diagram of size  $m$  and whose only IE-vector has  $m$  nonzero entries. Indeed, let  $K = \{\vartheta_1, \vartheta_2, \dots, \vartheta_m\}$  be a simplicial complex over  $[n]$  such that  $[n] = \bigcup K$  and  $|K| = m$ . We define  $F_i = \{t \in [m] : i \in \vartheta_t\}$  for  $1 \leq i \leq n$  and put  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ . It can easily be checked that  $\mathcal{V}(\mathcal{F}) = \mathcal{N}(\mathcal{F}) = K$  so, as observed in Example 2.3, the matrix  $A$  is square, lower-triangular, and has 1's on the diagonal; there is therefore a unique IE-vector for  $\mathcal{F}$  and it has  $m$  nonzero entries. In this section we improve this lower-bound.

We recall that by Corollary 2.4, every set system  $\mathcal{F}$  has a unique IE-vector with support in the Venn diagram  $\mathcal{V}(\mathcal{F})$ . This leads to the following observation.

**Lemma 4.1.** *Let  $\mathcal{F}$  be a finite family of sets with Venn diagram  $\mathcal{V}$ . If  $\mathcal{V} \cup \{[n]\}$ , considered as a poset with respect to the inclusion relation, is a join-semilattice,<sup>5</sup> then among all IE-vectors for  $\mathcal{F}$ , the one with support in  $\mathcal{V}$  has minimal  $\ell_1$ -norm.*

*Proof.* Let  $A$  be the matrix with rows indexed by  $\mathcal{V}$  and columns indexed by  $\mathcal{N} = \mathcal{N}(\mathcal{F})$ , as defined before Lemma 2.1, and let  $B$  be the  $m \times m$  submatrix consisting of the first  $m$  columns of  $A$ .

We want to show that every column of  $A$  is also a column of  $B$ . By the definition of  $A$ , this means that for every  $\sigma \in \mathcal{N}$  we need to find some  $\nu \in \mathcal{V}$  such that  $\{\tau \in \mathcal{V} : \sigma \subseteq \tau\} = \{\tau \in \mathcal{V} : \nu \subseteq \tau\}$ . It is easily seen that the join of all  $\tau \in \mathcal{V}$  with  $\tau \subseteq \sigma$  is such a  $\nu$  (we note that  $\nu \in \mathcal{V}$ , since all inclusion-maximal elements of  $\mathcal{N}$  are in  $\mathcal{V}$ ).

Hence every column of  $A$  occurs in  $B$  as asserted. It follows that every solution of  $A\mathbf{x} = \mathbf{1}$  can be transformed to a solution of  $B\mathbf{y} = \mathbf{1}$  with the same or smaller  $\ell_1$ -norm (if  $k$  is the index of a column outside  $B$  with  $x_k \neq 0$ , and that  $k$ th column equals the  $j$ th column of  $B$ , then we can zero out  $x_k$  while replacing  $x_j$  with  $x_j + x_k$ ). Since  $B\mathbf{y} = \mathbf{1}$  has a unique solution, it has to be a solution of minimum  $\ell_1$ -norm as claimed.  $\square$

We can now show that for arbitrary large  $n$  there exist families of  $n$  sets with Venn diagrams of size  $n$  for which any IE-vector has  $\ell_1$ -norm at least  $\left(\frac{n}{2}\right)^{3/2}$ .

*Proof of Theorem 1.2.* Let  $s = q^2 + q + 1$ , where  $q$  is a power of a prime number. Let  $(P, \mathcal{L})$  be a finite projective plane<sup>6</sup> of order  $q$ , and let us put  $S = P \cup \mathcal{L}$ .

We number the elements of  $P$  arbitrarily as  $P = \{p_1, p_2, \dots, p_s\}$ , and similarly  $\mathcal{L} = \{\ell_1, \ell_2, \dots, \ell_s\}$ . For  $i \in [s]$  we set  $F_i = \{\ell_i\} \cup \{p_j : p_j \in \ell_i\}$ , that is the line  $\ell_i$  together with all the points it contains, and  $F_{i+s} = \{p_i\}$ . Our set system is  $\mathcal{F} = \{F_1, \dots, F_n\}$ ,  $n = 2s$ .

To describe the Venn diagram  $\mathcal{V}(\mathcal{F})$ , we note that each line  $\ell_i \in \mathcal{L}$  is contained only in  $F_i$ , while each point  $p_i \in P$  is contained in  $F_{i+s}$  and in every  $F_{i'}$  with  $p_i \in \ell_{i'}$ . Therefore,  $\mathcal{V}$  consists of  $\tau_i = \{i\}$  and  $\tau_{i+s} = \{i+s\} \cup \{i' : p_i \in \ell_{i'}\}$ ,  $i \in [s]$ . In particular,  $m = |\mathcal{V}| = 2s = n$ . It is easy to find the unique IE-vector for  $\mathcal{F}$  with support in  $\mathcal{V}$ : the nonzero components are  $\alpha_k = 1$  for  $k = 1, 2, \dots, s$  and  $\alpha_k = -q$  for  $k = s+1, s+2, \dots, 2s$ . Thus  $\|\alpha\|_1 = s(q+1) \geq (q^2 + q + 1)^{3/2} = \left(\frac{n}{2}\right)^{3/2}$ .

<sup>5</sup>This means that for every  $\tau_1, \dots, \tau_k \in \mathcal{V}$  there is the join  $\tau = \bigvee_{i=1}^k \tau_i \in \mathcal{V} \cup \{[n]\}$ , meaning that all  $\tau_i \subseteq \tau$ , and also  $\tau \subseteq \tau'$  whenever  $\tau' \in \mathcal{V} \cup \{[n]\}$  contains all of  $\tau_1, \dots, \tau_k$ .

<sup>6</sup>A finite projective plane of order  $q$  is a pair of sets  $(P, \mathcal{L})$  where  $P$  is a set of  $q^2 + q + 1$  points and  $\mathcal{L} \subset 2^P$  is a set of  $q^2 + q + 1$  lines such that every line contains  $q+1$  points, every point is in  $q+1$  lines, every two lines intersect in a single point and every two points are contained in exactly one line. It is well known that a projective plane of order  $q$  exists whenever  $q$  is a power of a prime number.

It remains to check that  $\mathcal{V} \cup \{[n]\}$  is a join-semilattice; then Theorem 1.2 will follow from Lemma 4.1. Since  $\mathcal{V}$  is finite, it is enough to verify that the join  $\tau_i \vee \tau_j$  exists for every two  $i, j \in [n]$ ,  $i < j$ . If  $i, j \leq s$ , then  $\tau_i \cup \tau_j = \{i, j\}$ , and this is contained only in  $[n]$  and in  $\tau_{k+s}$ , where  $p_k$  is the point of intersection of  $\ell_i$  and  $\ell_j$ . Therefore,  $\tau_i \vee \tau_j = \tau_{k+s}$ . If  $i \leq s$  and  $j > s$ , then either  $\tau_i \subseteq \tau_j$  (which implies  $\tau_i \vee \tau_j = \tau_j$ ) or  $\tau_i \cup \tau_j$  has at least  $q + 3$  elements (which implies  $\tau_i \vee \tau_j = [n]$  since  $\mathcal{V}$  contains only sets of size at most  $q + 2$ ). Finally, if  $i, j > s$ , then  $\tau_i \cup \tau_j$  again has at least  $q + 3$  elements, implying  $\tau_i \vee \tau_j = [n]$ . This concludes the proof.  $\square$

## 5 Open problems

For several NP-hard problems, the best exponential-time algorithms rely on the inclusion-exclusion principle [BHK09, vRNvD09, NvR10]. Whether these algorithms can be improved using Theorem 1.1 is an open problem that is perhaps best illustrated on an example.

Consider for instance the question of counting the number of  $k$ -covers: given a family  $\mathcal{X} = \{X_1, X_2, \dots, X_p\}$  of subsets of  $[n]$ , we want to determine how many  $k$ -element subsets of  $\mathcal{X}$  have their union equal to  $[n]$ . Björklund et al. [BHK09, Section 3.1] proposed the following approach. For  $i \in [n]$ , let  $F_i$  denote the set of  $k$ -element subsets of  $\mathcal{X}$  whose union does not contain  $i$ :  $F_i = \{(Y_1, Y_2, \dots, Y_k) \in \mathcal{X} : i \notin Y_1 \cup Y_2 \cup \dots \cup Y_k\}$ . For a subset  $\sigma \subseteq [n]$  let  $\text{av}(\sigma) = \{X \in \mathcal{X} : X \cap \sigma = \emptyset\}$ . The number  $c_k$  of  $k$ -covers of  $\mathcal{X}$  can be written, using the inclusion-exclusion principle, as

$$c_k = |\mathcal{X}|^k - \left| \bigcup_{i=1}^n F_i \right| = |\mathcal{X}|^k - \sum_{\emptyset \subsetneq \sigma \subseteq [n]} (-1)^{|\sigma|+1} \left| \bigcap_{i \in \sigma} F_i \right| = |\mathcal{X}|^k - \sum_{\emptyset \subsetneq \sigma \subseteq [n]} (-1)^{|\sigma|+1} |\text{av}(\sigma)|^k. \quad (7)$$

Let  $f$  denote the indicator function of  $X$  and  $\tilde{f}$  its *Möbius* transform: for  $I \subseteq [n]$ ,  $\tilde{f}(I) = \sum_{S \subseteq I} f(S)$  ( $\tilde{f}$  is sometimes also called the *Zeta* transform). Since  $|\text{av}(\sigma)| = \sum_{S \subseteq [n] \setminus \sigma} f(S) = \tilde{f}([n] \setminus \sigma)$ ,  $c_k$  can be deduced from the Möbius transform of  $f$  by summing its  $k$ th powers.

If  $\mathcal{K}$  is a simplicial complex with  $n$  vertices and  $|\mathcal{K}|$  simplices, then the values of  $\tilde{f}(\sigma)$  for all  $\sigma \in \mathcal{K}$  can be computed in  $O(n|\mathcal{K}|)$  time by Yates' algorithm [Knu97, Section 4.3.4]. The above method for counting  $k$ -covers therefore runs in time  $O(n2^n)$ . Simplifying the inclusion-exclusion formula (7) while keeping its support hereditary, as Theorem 1.1 does, improves the running time to  $O(ns)$ , where  $s$  is the size of the formula ( $s = m^{O(\log^2 n)}$  in Theorem 1.1). When the Venn diagram of the  $F_i$ 's has size  $m = 2^{o(n)}$ , this complexity becomes subexponential in  $n$ . However, the catch is that, in the above example and many other problems [BHK09, vRNvD09, NvR10], the family  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  is not standardized, which is a crucial assumption for the computational statement in Theorem 1.1. Whether a simplified formula can be computed efficiently in this context is an open problem.

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